e content for students of patliputra university

B. Sc. (Honrs) Part 1 paper 1

Subject:Mathematics

Title/Heading of topic: Hyperbolic function

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Hyperbolic Function

def : **hyperbolic cosine (cosh) :**

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

hyperbolic sine (sinh) :

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$$

$$\sinh = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$$

For each real **t** $\cos^2 t + \sin^2 t = 1$ thus the point **(cos t, sin t)** lies on the unit circle : $x^2 + y^2 = 1$.
Hence **sin** and **cos** are called "**circular functions**"

Similarly for real **t** $\mathrm{cohs}^2\ t - \sinh^2 t = 1$ thus the point (**cosh t, sinh t**) lies on the hyperbola : $x^2 - y^2 = 1$.
Hence **sinh** (cinch) and **cosh** (kosh) are called **"hyperbolic functions"**

Other hyperbolic functions:

Analogous to circular functions) :

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^{x} + e^{-x})}{(e^{x} - e^{-x})}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\coth x = \frac{1}{\tanh x}$$

Graphs of cosh x and sinh x

$$\cosh x = \frac{1}{2}(e^{x} + e^{-x}) \quad , \quad \sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{(e^{x} + e^{-x})}$$

$$\cosh 0 = \frac{1}{2}(e^{0} + e^{-0}) = \frac{1}{2}(1 + 1) = 1$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{(e^{x} - e^{-x})}$$

$$\sinh 0 = \frac{1}{2}(e^{0} - e^{-0}) = \frac{1}{2}(1 - 1) = 0$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

Inverse Hyperbolic Functions:

(Analogous to $\sin^{-1} x$ and $\cos^{-1} x$)

Logarithmic Forms:

$$y = \sinh^{-1} x \text{ means } x = \sinh y$$

$$\text{or} \quad x = \frac{1}{2}\left(e^{y} - e^{-y}\right)$$

$$\text{or } 2x = e^{y} - e^{-y}$$

multiply by e^{y} and rearrange to $e^{2y} - 2xe^{y} - 1 = 0$

Set $e^{y} = u$, so $e^{2y} = u^2$, to get

$u^2 - 2xu - 1 = 0$ (quadratic in u)

Roots:

$$u = e^{y} = \frac{2x + - \sqrt{4x^2 + 4}}{2}$$

$$= x + - \sqrt{x^2 + 1}$$

$-$ sign is rejected since $e^{y} > 0$

so $e^{y} = x + \sqrt{x^2 + 1}$

Taking $\ln$ and noting $\ln e = 1$

$$y = \sinh^{-1} x = \ln\left[x + \sqrt{x^2 + 1}\right] \quad \text{for all } x.$$

Similarly

$$\cosh^{-1} x = \ln\left[x + \sqrt{x^2 - 1}\right] \quad x \geq 1$$

Example: $y = \tanh^{-1} x$

$$x = \tanh y = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1},$$

hence $x\,(e^{2y} + 1) = e^{2y} - 1$,

$$\text{so} \quad e^{2y} = \frac{1 + x}{1 - x}$$

Since $e^{2y} > 0$ so $-1 < x < 1$ or $|x| < 1$. Taking $\ln$ we get

$$2y \ln e = \ln \frac{1 + x}{1 - x}$$

$$\text{or} \quad y = \tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}, \quad |x| < 1$$

Hyperbolic Identities :

From $\cosh x = \frac{1}{2}(e^x + e^{-x})$

and $\sinh x = \frac{1}{2}(e^x - e^{-x})$

we get $\cosh x + \sinh x = e^x$

and $\cosh x - \sinh x = e^{-x}$

Multiply last two equations to get

$$\cosh^2 x - \sinh^2 x = 1$$

$$(\cos^2 x + \sin^2 x = 1)$$

From this we can get others

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$(\sec^2 x = 1 + \tan^2 x)$$

$$\operatorname{cosech}^2 x = \coth^2 x - 1$$

$$(\operatorname{cosec}^2 x = 1 + \cot^2 x)$$

Now from definition;

$$\cosh(x+y) = \frac{1}{2}(e^{x+y} + e^{-(x+y)})$$

$$= \frac{1}{2}[e^x e^y + e^{-x} e^{-y}]$$

$$= \frac{1}{2}[\ (\cosh x + \sinh x)(\cosh y + \sinh y) + (\cosh x - \sinh x)(\cosh y - \sinh y)\ ]$$

or $$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$(\cos(x+y) = \cos x \cos y - \sin x \sin y)$$

Setting $y = x$ gives

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$(\cos 2x = \cos^2 x - \sin^2 x)$$

From this it is easy to show :

$$\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

$$(\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x)$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$(\sin(x+y) = \sin x \cos y + \cos x \sin y)$$

Similarly

$$\sinh 2x = 2\sinh x \cosh x$$

$$(\sin 2x = 2\sin x \cos x)$$

And

$$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} \quad \text{etc}$$

Recall Euler's Relation

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

By adding and subtracting

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

Now $\cosh x = \frac{1}{2}(e^x + e^{-x})$ for $x = j\theta$ one gets

$$\cosh j\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) = \cos\theta$$

or

$$\cosh j\theta = \cos\theta \quad \ldots\ldots\ldots (i)$$

Similarly $\sinh x = \frac{1}{2}(e^x - e^{-x})$

gives

$$\sinh j\theta = \frac{1}{2}(e^{j\theta} - e^{-j\theta}) = j\sin\theta$$

Hence

$$\sinh j\theta = j\sin\theta \quad \ldots\ldots\ldots (ii)$$

Put $\theta = jx$ in (i) to get

$$\cos jx = \cosh j^2 x = \cosh(-x) = \cosh x$$

or $\cos jx = \cosh x \ldots\ldots\ldots\ldots (iii)$

Put $\theta = jx$ in (ii) to get

$$j \sin jx = \sinh j^2 x = \sinh(-x) = -\sinh x$$

Hence $j \sin jx = j^2 \sinh x$

or $\sin jx = j \sinh x \ldots\ldots\ldots\ldots (iv)$

Important Results :

$\sin jx = j \sinh x$

$\sinh jx = j \sin x$

$\cos jx = \cosh x$

$\cosh jx = \cos x$

$\tan jx = j \tanh x$

$\tanh jx = j \tan x$

Example : solve $\sin z = -\frac{5}{4}$ NB. z cannot be real

Let $z = x + jy$, then

$$\sin(x + jy) = \sin x \cos jy + \cos x \sin jy = \sin x \cosh y + j \cos x \sinh y$$

Hence $\sin z = \sin(x + jy)$

$$= \sin x \cosh y + j \cos x \sinh y$$

$$= -\frac{5}{4}$$

Thus $\sin x \cosh y = -\frac{5}{4}$ ....(i)

$\cos x \sinh y = 0$. ....(ii)

From (ii) either

$\sinh y = 0$ ie $y = 0$, or

$\cos x = 0$, $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

(a) If $y = 0$ then $\cosh y = 1$ and from (i) $\sin x = -\frac{5}{4}$

Do you believe that?

(b) If $\cos x = 0$, then $\sin x = +1$ or -1 (why?)

(i) $\sin x = +1$, then from (i) $\cosh y = -\frac{5}{4}$ (Impossible)

(ii) $\sin x = -1$, ie. $x = \ldots -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

Also now from (i)

$$\cosh y = \frac{5}{4}$$

or $\frac{1}{2}(e^{y} + e^{-y}) = \frac{5}{4}$,

or $2e^{y} + 2e^{-y} - 5 = 0$

or $2e^{2y} - 5e^{y} + 2 = 0$

A quadratic in e^{y} with factors

$$(2e^{y} - 1)(e^{y} - 2) = 0$$

Roots: $e^{y} = \frac{1}{2}$ and $e^{y} = 2$

Taking ln: $y = \ln\left(\frac{1}{2}\right)$ or $y = -\ln 2$

and $y = \ln 2$.

Hence $z = x + yj$ $x = \ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

$y = \pm\ln 2$